Problem Set 1

This problem set is in two parts. The first part is some mathematical preliminaries to refresh your math knowledge. The second part is related to the material that we discussed in class.

Part I. Math

Problem 1. Exercise 3.3.c,d, Exercise 3.4.e,f and Exercise 3.10.a, b in SLP.

Problem 2.

a. Suppose $p \geq 1$ is a real number and consider the following space and norm

$$L^p = \left\{ f(x) | \, f : [0,1] \to \mathbb{R}, \int_0^1 |f(x)|^p \, dx < \infty \right\}$$
$$\|f\|_p = \left(\int_0^1 |f(x)|^p \, dx \right)^{\frac{1}{p}}, \forall f \in L^p$$

$\|\cdot\|_p$ is called the p-norm. Show that $\left(L^p, \|\cdot\|_p\right)$ is a normed vector space.

b. Another way to make the space of functions into a normed vector space is as follows:

$$L^\infty = \left\{ f(x) \, | f : [0,1] \to \mathbb{R}, \sup_{x \in [0,1]} |f(x)| < \infty \right\}$$
$$\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|, \forall f \in L^\infty$$

Show that $(L^\infty, \|\cdot\|_\infty)$ is complete.

c. Show that the normed vector space given in exercise 3.4.f is not complete. You have to provide an example. Show why that example does not work for 3.4.e.

Problem 3. Consider the functional equation:

$$f(x) = \beta \left((u(x))^{\frac{1}{\alpha}} + \left(\int_a^b \phi(x,t) \, f(t) \, dt \right)^{\frac{1}{\alpha}} \right)^\alpha$$

where in the above, $u(x) : [a,b] \to \mathbb{R}_+$ is a bounded function and $\alpha \geq 1 > \beta$. Furthermore, $\int_a^b \phi(x,t) \, dt = 1$, for all $x \in [a,b]$. Show that there exists a unique $f : [a,b] \to \mathbb{R}_+$ which satisfies the above equation.

Hint: Use contraction mapping theorem.

Part II

Problem 4. Sequential Trading vs Time-0 Trading Here you are asked to show that if we allow for sequential trading in the model in class, the outcome will be the same. Suppose that the agents, preferences and technology are the same as those discussed in class.

Trading is organized as follows: in each period, t, households can purchase any amount of a financial asset called IOU - also known as an *Arrow Security* - which entitles them to a unit of the consumption good in period $t+1$. Thus, their budget constraint is given by

$$p_{c,t}c_t^j + p_{x,t}x_t^j + q_t a_t^j \leq w_t\left(e_t^j - \ell_t^j\right) + r_t k_t^i + p_{c,t}a_{t-1}^i$$

where q_t is the price of Arrow securities. To be more precise, there are 5 spot markets now that open each day: markets for consumption goods, investment goods, labor, rental of capital, IOUs. Now, each households maximizes his/her utility subject to the above budget constraint and another constraint of the form

$$a_t \geq -A$$

where A is some large positive number.

(**Side Note:** The constraint above requires some explanation. The issue is that if the household can freely borrow, he/she might run ponzi schemes. This is not an issue if the horizon was finite since there is always a last period in which the household has to pay up. With infinite horizon, and without any constraint, in theory it is possible to have a ponzi scheme: A scheme in which the household borrows and then pays the interest and the principal with new borrowing and does this forever. Somehow we need to impose a constraint so that such ponzi schemes are not possible. The above is one. The typical way of doing it is to impose the following constraint

$$\lim_{t\to\infty} q_t a_t^j \geq 0$$

i.e., the total market value of your assets have to be finite. Note also that at the optimum this constraint wont be binding so we dont have to worry about it!)

a. What should be the feasibility constraint in this new economy? In other words, what is market clearing for the market for Arrow securities? Explain your answer

b. Define a competitive equilibrium for this economy.

c. Show that any allocation resulting from any competitive equilibrium in the Arrow-Debreu economy, i.e., with time-0 trading, is also part of an allocation in the CE of the economy with sequential markets.

d. Consider a competitive equilibrium in the sequential market economy described above. What is the relationship between prices of investment good, $p_{x,t}$, and rental rate of capital, r_t? What is the relationship between rental rate of capital, r_t, and the price of arrow securities, q_t? *Hint:* Think about arbitrage!!!

Problem 5. Intermediate Goods and Aggregate Production Function As we mentioned in class, we assumed that only inputs into production are capital and labor. Here, we study what happens when there are intermediate goods. Consider a static economy where there are N goods that can be used in both consumption and production. Moreover, suppose that production also uses capital and labor.

There are I households who have preferences given by

$$U\left(\{c_i\}_{i=1}^N\right) = \sum_{i=1}^N \alpha_i \log c_i$$

where c_i is consumption of good i by each households. Additionally, each household owns $\frac{\overline{K}}{I}$ unit of capital and $\frac{\overline{L}}{I}$ units of labor.

Each good i is produced by a representative firm in industry i according to the production function

$$Y_i = A_i K_i^{\alpha} L_i^{\gamma-\alpha} \prod_{j=1}^N Q_{ij}^{(1-\gamma)\omega_{ij}}$$

where Q_{ij} is the total quantity of good j used in the production of i; K_i, L_i are capital and labor used in the production of good i while ω_{ij} satisfies

$$\sum_{j=1}^N \omega_{ij} = 1$$

Each representative firm in industry i takes as given the prices of all goods and services. Note that we can think about the weights ω_{ij} as the network structure of production in the economy. It determines which sectors sell to which and at what intensity.

There is a market for each good and for capital and labor services. Let price of good i be given by p_i while r denotes rental rate of capital and w denotes the wage.

a. Define a competitive equilibrium for this economy.

b. In what follows, we first try to solve for prices in terms of rental rate of capital and wage. To do this, write the cost minimization problem of the firm:

$$C_i(Y) = \min rK_i + wL_i + \sum_{j=1}^N p_j Q_{ij}$$

subject to

$$A_i K_i^{\alpha} L_i^{\gamma-\alpha} \prod_{j=1}^N Q_{ij}^{(1-\gamma)\omega_{ij}} = Y$$

Show that the above cost function has the form $C_i(Y) = \psi_i Y$. Find the formula for ψ_i in terms of prices of all goods and

c. Show that in equilibrium, it must be that $p_i = \psi_i$. Explain Why.

d. Using the above set of equations, $p_i = \psi_i$, solve for the vector of prices of goods as a function of rental rate of capital, wage, and productivities.

e. By Walras' law, we can always normalize one price to 1. Suppose we set $w = 1$. Using market clearing conditions on capital, labor and goods, solve for the entire vector of prices, and production in each sector and rental rate of capital.

f. Using your solution above, write aggregate GDP in terms of capital and labor. Recall that intermediate inputs are not counted in GDP.

g. Does the network structure of production affect the aggregate production function? To answer this, consider a one-percent change in productivity of each sector (a one-percent increase in A_i) and describe how aggregate output changes in precentages. What determines the magnitude of the response? How are other industries affected by this shock? Provide an intuition for your answer.

h. In order to understand the effect of the network structure, compare your answer for part f for two networks: a production chain and a star! A production chain is a network structure in which:

$$\omega_{ij} = \begin{cases} 1 & i \geq 1, j = i - 1 \\ 0 & \text{otherwise} \end{cases}$$

This network structure resembles a chain where sector i uses output of sector $i-1$ as their only intermediate input while sector 1 uses its own output as an input.
A star, on the other hand, is a network structure in which

$$\omega_{ij} = \begin{cases} 1 & i \geq 1, j = 1 \\ 0 & \text{otherwise} \end{cases}$$

This network structure resembles a star where every other sector uses the output of sector 1. How does the structure of the network affect the aggregate production function in these two economies? What can you say about the importance of each sector and their effect on aggregate production under each structure.
You need to describe the effect of a shock in a sector on the aggregate output and outputs of other industries.

i. In this part, you will try to take this model to the data. Go to http://www.bea.gov and under interactive data, find the input-output table data. The table entitled "The Use of Commodities by Industries, Before Redfinitions" provides information about how much each industry spends on other industries. Now consider the model above. Use 2016 the "Use Table" to calibrate the values of ω_{ij} as defined above. Provide a reasoning. In your estimated ω_{ij}'s, what pattern stands out?

Problem 6. Introducing Health Status in Our Model Consider the model discussed in class and assume that households utility functions are additively separable over time and depend on consumption and leisure. Suppose that we want to modify the model to introduce health status and health expenditure in this model. Describe various ways that an individual's health status

and expenditure can affect the commodity space, households' preferences, budget constraint and firm's production function. Write down a model capturing your intuition and define a competitive equilibrium for this economy.

Problem 7. The Economy and the Environment. In our discussion of the macroeconomic model developed in class, we ignored environmental considerations. How do these considerations affect the model. Write down an extension of our model where environmental factors interact with the economy. Define a competitive equilibrium. Does the first welfare theorem hold in your model? Why or why not?